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Trajectories and spin motion of massive spin-$\frac{1}{2}$ particles in gravitational fields

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Abstract. From covariant Dirac theory in curved space–time, dynamical equations for the motion of the spin and the spin-induced non-geodesic behaviour of the particle trajectories are deduced. This is done for arbitrary space–times in a generally covariant and observer-independent way. The procedure is thereby based on a WKB scheme and a Gordon decomposition of the Dirac probability four-current. A complete correspondence between the quantum mechanical equations of motion and the classical equations for extended isolated bodies or pole–dipole particles is found. This can as well be taken as a confirmation that to the first order of a WKB approximation the gyro-gravitational factors of the classical angular momentum and of the intrinsic quantum mechanical spin agree.

1. Introduction

With regard to an external electromagnetic field in flat space–time Bargmann *et al* (1959) derived within the framework of classical mechanics and electrodynamics the classical motion of charged spinning particles and an equation for the precession of the polarisation four-vector. Later it has been possible successfully to deduce these classical equations directly from quantum mechanics in discussing the WKB solutions to the Dirac equation in the classical limit as $\hbar$ approaches zero. For a derivation and the earlier literature see Rafanelli and Schiller (1964).

In a metric theory of gravitation the interaction with an external gravitational field is described by embedding physics in an otherwise-determined curved space–time. In curved space–time as well we have equations of motion for classical spinning bodies (extended bodies or pole–dipole point particles) and equations for the precession of their classical angular momentum. The question also arises for external gravitational fields: Is it possible without restricting the space–time to derive from generally covariant Dirac theory dynamical equations for the influence of the spin on particle trajectories and for the motion of the spin along these trajectories, and do these equations 'mirror' the classical equations? It is the purpose of this paper to demonstrate that this can be done in a satisfactory way by means of a WKB extension using a Gordon decomposition of the Dirac probability four-current.

Because of some misleading remarks in the literature, it should be stressed, however, that it cannot be the intention to 'deduce' the classical equations of motion for macroscopic bodies from quantum mechanics in discussing some limiting case. There is no conceptually and mathematically elaborated scheme which would enable this to be done. The only applications of quantum mechanics which are of a genuine macroscopic nature can be found in connection with Bose–Einstein condensation for superfluids and similar phenomena. Coherent states should be as suitable for discussing a reasonable approach to classical physics. In any case, the quantum mechanical spin itself always remains as an intrinsic quantity without any classical limit. Also, an attempt to complete this intrinsic spin and to start instead from the total angular momentum to derive classical physics would have no advantage over purely classical calculations, because the same difficulties (such as the definition of a mass centre or of multiple moments) would soon appear.

Nevertheless, it is intrinsically important to study within quantum mechanics the coupling of the spin to the gravitational field, the spin motion and its influence on trajectories. The common group-theoretical origin of angular momentum and spin then suggests a comparison of the quantum mechanical equations with those describing the influence of classical angular momentum in curved space–time. Similarities are to be expected.

It is well known that the existence of negative-energy solutions makes it difficult to relate the dynamical variables of the c-number Dirac theory to the corresponding classical quantities. An instructive example is the *Zitterbewegung.* It vanishes if the state in question is a superposition of positive-energy waves only. Therefore to establish a correspondence with classical mechanics one has to restrict the influence of the negative-energy solutions and proceed to a classical limit. By the WKB expansion this can be done in an observer-independent way. In addition, to reduce the total angular momentum to the spin only we have to use a localised approach which refers to quantities defined on a worldline and its infinitesimal surrounding. After the introduction of an observer field they would obtain the meaning of physical densities.

To enable a comparison with the dynamics of isolated classical bodies with multipole moments we note that the equations of motion for the monopole and dipole moments of a body as determined by the energy–momentum tensor are

$$\mathrm{d}P^{\alpha}/\mathrm{d}s = \tfrac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}V^{\beta}S^{\gamma\delta} \tag{1.1a}$$

$$\mathrm{d}S^{\alpha\beta}/\mathrm{d}s = P^{\alpha}V^{\beta} - P^{\beta}V^{\alpha} \tag{1.1b}$$

where the influence of higher moments is neglected. For the respective derivations, difficulties and further references see Dixon (1979). P^{α} is thereby the dynamically-defined momentum vector and $S^{\alpha\beta}$ the angular momentum tensor of the body. The kinematically introduced velocity $V^{\alpha} = \mathrm{d}z^{\alpha}/\mathrm{d}s$ is the tangent vector to the time-like worldline $z^{\alpha}(x)$ to which the moments refer. It is usually taken to be that of a suitably defined mass centre. The kinematical velocity V^{α} and the dynamical momentum P^{α} will not in general be parallel.

The same equations (1.1) are obtained for a localised classical particle with an internal structure. The corresponding energy–momentum tensor is in this case given by a pole–dipole-like mass distribution. It contains a delta function and its first derivatives. The support of both is the V^{α} worldline. For the calculation and further references see Westpfahl (1969).

Because several authors have already taken an interest in the problem we are studying we give a brief discussion of the previous literature in the Appendix.

2. WKB expansion

It is convenient to formulate the Dirac theory in curved space–time with respect to an orthonormal tetrad field† $h^{\alpha}_{a}(x)$:

$$h^{\alpha}_{a}h^{\beta}_{b}g_{\alpha\beta}=\eta_{ab}. \tag{2.1}$$

The Dirac equation then takes the form

$$\mathrm{i}\gamma^{\mu}\Psi_{;\mu}-(m/\hbar)\Psi=0 \tag{2.2}$$

with

$$\gamma^{\mu}=h^{\mu}_{a}\gamma^{a}, \qquad \gamma^{(a}\gamma^{b)}=\eta^{ab}, \qquad \gamma^{(\mu}\gamma^{\nu)}=g^{\mu\nu}. \tag{2.3a, b, c}$$

The covariant spinor derivative is thereby given by

$$\Psi_{;\mu}=\Psi_{,\mu}+\Gamma_{\mu}\Psi, \qquad \Gamma_{\mu}=\tfrac{1}{4}h^{\alpha}_{a;\mu}h_{\alpha b}\gamma^{b}\gamma^{a} \tag{2.4a, b}$$

and for the gamma matrices we have

$$\gamma^{\alpha}{}_{;\epsilon}=0, \qquad \gamma^{a}{}_{,\epsilon}=0. \tag{2.5a, b}$$

The γ^{a} are the standard Dirac matrices.

In the WKB expansion the Dirac solutions $\Psi(x)$ are written as a phase factor and a four-spinor amplitude which is a power series in $\hbar$:

$$\Psi(x)=\exp(\mathrm{i}S(x)/\hbar)\sum_{n=0}^{\infty}(-\mathrm{i}\hbar)^{n}a_{n}(x). \tag{2.6}$$

WKB solutions of the Dirac equation to any given order in $\hbar$ are obtained by inserting equation (2.6) in equation (2.3) and equating to zero the coefficient of each power of $\hbar$. From the coefficients of $\hbar^{0}$ and $\hbar^{1}$ we obtain equations for the first terms of the spinor amplitude:

$$(\gamma^{\alpha}S_{,\alpha}+m)a_{0}=0, \tag{2.7}$$

$$(\gamma^{\alpha}S_{,\alpha}+m)a_{1}=-\gamma^{\alpha}a_{0;\alpha}. \tag{2.8}$$

The condition that equation (2.7) has a non-trivial solution is

$$\det(\gamma^{\alpha}S_{,\alpha}+m)=0. \tag{2.9}$$

This implies the Hamilton–Jacobi equations of relativistic spinless particles

$$S^{,\alpha}S_{,\alpha}=m^{2} \tag{2.10}$$

which can as well be obtained by iterating equation (2.7) using equation (2.3c). We define

$$p_{\alpha}=-S_{,\alpha}, \qquad p^{\alpha}p_{\alpha}=m^{2} \tag{2.11}$$

and introduce the corresponding normalised time-like vector u^{α}

$$u_{\alpha}=(1/m)p_{\alpha}=-(1/m)S_{,\alpha}, \qquad u^{\alpha}u_{\alpha}=1. \tag{2.12}$$

† $c=1$. The signature of the metric tensor $g_{\alpha\beta}$ is $(---+)$. $;\alpha$ denotes the covariant and $,\alpha$ the partial derivative. $\alpha,\beta,\ldots=1,\ldots,4$ are tensor indices raised and lowered with $g_{\alpha\beta}$. $a,b,\ldots=1,\ldots,4$ and $\hat{a},\hat{b},\ldots=1,2,3$ are tetrad indices raised and lowered with $\eta_{ab}=\mathrm{diag}(-1,-1,-1,+1)$. The corresponding object is a Riemann scalar with regard to $a,b,\ldots$. Particular values of $a,b\ldots$ are denoted by brackets: $A^{(1)}=A^{a=1}$. Symmetrisation: $A_{(\alpha\beta)}=\frac{1}{2}(A_{\alpha\beta}+A_{\beta\alpha})$. Antisymmetrisation: $A_{[\alpha\beta]}=\frac{1}{2}(A_{\alpha\beta}-A_{\beta\alpha})$.

u^α is the tangent vector to the trajectories orthogonal to the system of space-like hypersurfaces $S=$ constant. Accordingly the trajectories form a congruence of time-like worldlines which is geodesic:

$$\dot{u}_\alpha = u_{\alpha;\epsilon}u^\epsilon = 0 \tag{2.13}$$

and rotation-free:

$$\omega_{\alpha\beta} = u_{[\alpha;\beta]} - \dot{u}_{[\alpha}u_{\beta]} = 0. \tag{2.14}$$

The remaining differential behaviour of the congruence is given by

$$u_{\alpha;\beta} = \hat{\sigma}_{\alpha\beta} + \theta h_{\alpha\beta}/3, \qquad u_{[\alpha;\beta]} = 0 \tag{2.15a, b}$$

with

$$h_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta, \qquad h_{\alpha\beta}u^\beta = 0. \tag{2.16}$$

It still depends on the shear $\hat{\sigma}_{\alpha\beta}$

$$\hat{\sigma}_{\alpha\beta} = h^\kappa_\alpha h^\lambda_\beta u_{(\kappa;\lambda)} - u^\epsilon{}_{;\epsilon}h_{\alpha\beta}/3 \tag{2.17}$$

$$\hat{\sigma}_{[\alpha\beta]} = 0, \qquad \hat{\sigma}^\epsilon{}_\epsilon = 0, \qquad \sigma_{\alpha\beta}u^\alpha = 0 \tag{2.18}$$

and the expansion θ

$$\theta = u^\alpha{}_{;\alpha} \tag{2.19}$$

of the congruence. Both are determined by initial conditions and the metric properties of the space–time.

By equation (2.7) $a_0(x)$ is only determined algebraically. The corresponding matrix is of rank two. The general solution of equation (2.7) is therefore of the form

$$a_0(x) = \beta_1(x)b_{01}(x) + \beta_2(x)b_{02}(x) \tag{2.20}$$

where $b_{01}(x)$ and $b_{02}(x)$ are the two linearly independent solutions†

$$b_{01} = \left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix} 1 \\ 0 \\ \dfrac{k^{(3)}}{E+m} \\ \dfrac{k^{(1)}+ik^{(2)}}{E+m} \end{pmatrix} \qquad b_{02} = \left(\frac{E+m}{2m}\right)^{1/2}\begin{pmatrix} 0 \\ 1 \\ \dfrac{k^{(1)}-ik^{(2)}}{E+m} \\ -\dfrac{k^{(3)}}{E+m} \end{pmatrix} \tag{2.21}$$

with

$$E(x) = p^\mu h^{(4)}_\mu, \qquad k^{\hat{a}}(x) = p^\mu h^{\hat{a}}_\mu. \tag{2.22}$$

Beyond that the differential behaviour of $a_0(x)$ is additionally restricted by the solvability condition of equation (2.8). Equation (2.8) is an inhomogeneous linear algebraic equation for a_1. Accordingly the condition for the existence of a non-trivial

† We choose

$$\gamma^{\hat{a}} = \begin{pmatrix} 0 & \sigma^{\hat{a}} \\ -\sigma^{\hat{a}} & 0 \end{pmatrix} \qquad \gamma^{(4)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where the $\sigma^{\hat{a}}$ are the standard Pauli spin matrices.

solution a_1 is that all solutions of the corresponding transposed homogeneous equations are orthogonal to the inhomogeneity. Comparison with equation (2.7) shows that these solutions are in our case given by $\overline{b_{01}}$ and $\overline{b_{02}}$. The solvability conditions of equation (2.8) are therefore†

$$\overline{b_{01}}\gamma^\alpha a_{0;\alpha} = 0, \qquad \overline{b_{02}}\gamma^\alpha a_{0;\alpha} = 0. \tag{2.23}$$

3. Propagation equations

The equations (2.23) imply propagation equations for the spinor $a_0(x)$ along the u^α lines. To show this we restrict to an arbitrary but fixed worldline of the u^α congruence and choose a particular tetrad field on and in the neighbourhood of the worldline. Equations which are invariant with regard to local tetrad rotations coupled with spin transformations

$$h_a^\alpha(x) \to h_a'^\alpha(x) = \hat{\Omega}_a^b(x) h_b^\alpha(x), \qquad \chi(x) \to \chi'(x) = \hat{S}(x)\chi(x) \tag{3.1}$$

may then be verified without any loss of generality in using this particular tetrad field.

On this worldline itself we adjust the time-like vector $h_{(4)}^\alpha$ of the tetrad parallel to u^α:

$$h_{(4)}^\alpha \overset{*}{=} u^\alpha \tag{3.2}$$

and propagate as well the space-like vectors $h_{\hat{a}}^\alpha$ parallelly along the geodesic u^α worldline. Additionally we adjust the tetrad field in the neighbourhood of the worldline by parallel propagation in all directions, so that we have in total on the worldline

$$h_{\hat{a};\epsilon}^\alpha \overset{*}{=} 0. \tag{3.3}$$

In the following the asterisk denotes the worldline and the choice (3.3). For the spinor affinity this implies with equation (2.4*b*)

$$\Gamma_\alpha \overset{*}{=} 0. \tag{3.4}$$

The solutions b_{01} and b_{02} reduce because of

$$E \overset{*}{=} m, \qquad k^{\hat{a}} \overset{*}{=} 0 \tag{3.5}$$

to

$$b_{01} \overset{*}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad b_{02} \overset{*}{=} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

Thus we have with equations (2.3) and (3.2):

$$\overline{b_{01}}\gamma^\mu b_{01} = u^\mu, \qquad \overline{b_{02}}\gamma^\mu b_{01} = 0, \tag{3.7a}$$

$$\overline{b_{01}}\gamma^\mu b_{02} = 0, \qquad \overline{b_{02}}\gamma^\mu b_{02} = u^\mu. \tag{3.7b}$$

For the derivatives of the components we find

$$E_{,\alpha} h_b^\alpha = m(u^\epsilon h_\epsilon^{(4)})_{;\alpha} h_b^\alpha \overset{*}{=} 0, \tag{3.8}$$

† The bar denotes the adjoint: $\bar{\Psi} = \Psi^+ \gamma^{(4)}$.

$$k^{\hat{a}}{}_{,\alpha}h^{\alpha}_{(4)} = m(u^{\epsilon}h^{\hat{a}}_{\epsilon})_{;\alpha}h^{\alpha}_{(4)} \triangleq 0, \tag{3.9}$$

$$k^{\hat{a}}{}_{,\alpha}h^{\alpha}_{\hat{b}} = m(u^{\epsilon}h^{\hat{a}}_{\epsilon})_{;\alpha}h^{\alpha}_{\hat{b}} \triangleq m(\hat{\sigma}_{\epsilon\alpha}+\tfrac{1}{3}\theta h_{\epsilon\alpha})h^{\epsilon}_{\hat{a}}h^{\alpha}_{\hat{b}} \tag{3.10}$$

where we have used equations (3.3) and (2.15). Thence, and because of equation (2.18), we obtain from equations (2.21) and (3.6) after some calculation

$$\overline{b_{01}}\gamma^{\epsilon}b_{01,\epsilon} \triangleq \theta/2, \qquad \overline{b_{02}}\gamma^{\epsilon}b_{01,\epsilon} \triangleq 0, \tag{3.11a}$$

$$\overline{b_{01}}\gamma^{\epsilon}b_{02,\epsilon} \triangleq 0, \qquad \overline{b_{02}}\gamma^{\epsilon}b_{02,\epsilon} \triangleq \theta/2. \tag{3.11b}$$

Note that the shear $\hat{\sigma}_{\alpha\beta}$ and other derivatives orthogonal to the u^{α} congruence do not appear.

The relations above enable the evaluation of the solvability condition (2.23). Inserting equation (2.20) we first obtain

$$\beta_{1,\alpha}u^{\alpha} + \beta_1\overline{b_{01}}\gamma^{\epsilon}b_{01,\epsilon} + \beta_2\overline{b_{01}}\gamma^{\epsilon}b_{02,\epsilon} \triangleq 0, \tag{3.12a}$$

$$\beta_{2,\alpha}u^{\alpha} + \beta_1\overline{b_{02}}\gamma^{\epsilon}b_{01,\epsilon} + \beta_2\overline{b_{02}}\gamma^{\epsilon}b_{02,\epsilon} \triangleq 0 \tag{3.12b}$$

which reduce with (3.11) to the generally valid equations

$$\beta_{1,\alpha}u^{\alpha} = -(\theta/2)\beta_1, \qquad \beta_{2,\alpha}u^{\alpha} = -(\theta/2)\beta_2. \tag{3.13}$$

These describe the propagation of the scalar factors $\beta_1(x)$ and $\beta_2(x)$ and therefore the precession of the spin along the worldlines of the u^{α} congruence. Furthermore because of equations (3.4) and (3.6) we have

$$b_{01;\alpha}u^{\alpha} = 0, \qquad b_{02;\alpha}u^{\alpha} = 0. \tag{3.14}$$

These equations are first of all obtained on the worldline $*$ using the choice (3.3). But they are again generally valid for the whole congruence. From equations (3.13) and (3.14) we finally obtain as *propagation equation* for the spinor a_0

$$a_{0;\alpha}u^{\alpha} = -(\theta/2)a_0. \tag{3.15}$$

It proves to be convenient to introduce a normalised spinor b_0 proportional to a_0 with

$$a_0 = fb_0, \qquad \overline{b_0}b_0 = 1. \tag{3.16}$$

Because

$$\overline{a_0}a_0 = \beta_1^*\beta_1 + \beta_2^*\beta_2 = f^2 \tag{3.17}$$

we obtain with equation (3.13) the respective propagation equations

$$f_{,\alpha}u^{\alpha} = -(\theta/2)f \tag{3.18}$$

and

$$b_{0;\alpha}u^{\alpha} = 0 \tag{3.19}$$

which demonstrate that the spinor b_0 is parallelly propagated along the u^{α} congruence.

4. Gordon decomposition

To exhibit explicitly the influence of the spin on the trajectories, we perform a *Gordon decomposition* (Gordon 1928) of the Dirac probability current j^{α}:

$$j^{\alpha} = \bar{\Psi}\gamma^{\alpha}\Psi, \qquad j^{\alpha}{}_{;\alpha} = 0. \tag{4.1}$$

Introducing

$$\sigma^{\alpha\beta} = \mathrm{i}\gamma^{[\alpha}\gamma^{\beta]} \tag{4.2}$$

this leads to

$$j^{\alpha} = j_c^{\alpha} + j_M^{\alpha} \tag{4.3}$$

with the definitions

$$j_c^{\alpha} = (\hbar/2m\mathrm{i})(\bar{\Psi}^{;\alpha}\Psi - \bar{\Psi}\Psi^{;\alpha}) \tag{4.4}$$

and

$$j_M^{\alpha} = (\hbar/2m)(\bar{\Psi}\sigma^{\alpha\beta}\Psi)_{;\beta} \tag{4.5}$$

which are invariant under the coupled transformations (3.1). Both currents satisfy a continuity equation

$$j^{\alpha}_{c;\alpha} = 0, \qquad j^{\alpha}_{M;\alpha} = 0. \tag{4.6}$$

For a physical interpretation of the decomposition above and for the discussion of the results in §§ 5 and 6 below it is useful to recall that in $\Psi = \binom{\Psi_A}{\Psi_B}$ the components Ψ_B are of order v/c compared with Ψ_A. Thus we have for the bilinear densities

$$\bar{\Psi}\Psi = \Psi_A^+\Psi_A + \mathrm{O}(v^2/c^2), \qquad \bar{\Psi}\sigma^{\hat{a}(4)}\Psi = \mathrm{O}(v/c), \tag{4.7a, b}$$

$$\bar{\Psi}\sigma^{\hat{a}\hat{b}}\Psi = \Psi_A^+\sigma^{\hat{c}}\Psi_A + \mathrm{O}(v^2/c^2). \tag{4.7c}$$

As will be seen below, we also have

$$j_c^{\alpha}u_{\alpha} = j^{\alpha}u_{\alpha} + \mathrm{O}(\hbar^2). \tag{4.8}$$

Hence, and because the space-like part of j_c^{α} behaves with equation (4.7) as the three-vector current density in Schrödinger theory, j_c^{α} is interpreted as a convection four-current. With equation (4.7) we obtain the result that j_M^{α} represents the curl of the spin density. Because the spin of the electron is coupled to a magnetic dipole moment, this curl of a magnetic dipole density is, according to Maxwell theory, equivalent to an electric current. Thus with regard to external electromagnetic fields j_M^{α} has the meaning of a magnetisation current.

These physical interpretations suggest that an attempt to reflect the equations of motion (1.1) mentioned in the Introduction should be based on the streamlines of the convection current j_M^{α}.

Introducing the WKB expansion (2.6) we obtain with equation (3.16) up to terms of the order $\hbar^2$

$$j_c^{\alpha} = [f^2 + (\hbar/\mathrm{i})(\overline{a_0}a_1 - \overline{a_1}a_0)]u^{\alpha} + (\hbar/2m\mathrm{i})f^2(\overline{b_0}^{;\alpha}b_0 - \overline{b_0}b_0{}^{;\alpha}) + \mathrm{O}(\hbar^2) \tag{4.9}$$

and

$$j_M^{\alpha} = (\hbar/2m)(\overline{a_0}\sigma^{\alpha\beta}a_0)_{;\beta} + \mathrm{O}(\hbar^2) = j_{M1}^{\alpha} + \mathrm{O}(\hbar^2). \tag{4.10}$$

Note that the term in equation (4.9) which contains b_0 is orthogonal to u^{α}. This is an immediate consequence of equation (3.19). The same is the case for j_{M1}^{α}:

$$j_{M1}^{\alpha}u_{\alpha} = (\hbar/2m)(\overline{a_0}\sigma^{\alpha\beta}a_0)_{;\beta}u_{\alpha} = (\hbar/2m)(\overline{a_0}\sigma^{\alpha\beta}a_0u_{\alpha})_{;\beta} = 0 \tag{4.11}$$

where we have used (2.15b), the special choice of equations (3.2), (3.4), (3.6) and the standard γ^a matrices.

5. Spin motion

The convection current j_c^α of equation (4.4) defines a congruence of time-like curves with tangent vector v^α:

$$v^\alpha \sim j_c^\alpha, \qquad v^\alpha v_\alpha = 1, \tag{5.1}$$

which agrees to lowest order in $\hbar$ with the u^α congruence orthogonal to the hypersurfaces S = constant:

$$v^\alpha = u^\alpha + \mathrm{O}(\hbar). \tag{5.2}$$

We look at a solution $\Psi(x)$ of the 'free' Dirac equation (2.2) as describing a stream of particles. With regard to the spin of these particles we introduce

$$\frac{\bar{\Psi}\sigma^{\alpha\beta}\Psi}{\bar{\Psi}\Psi} = \overline{b_0}\sigma^{\alpha\beta}b_0 + \mathrm{O}(\hbar). \tag{5.3}$$

According to equation (4.7) we have to interpret $\bar{\Psi}\sigma^{\alpha\beta}\Psi$ as spin density and $\bar{\Psi}\Psi$ as the particle number density. Thus the components of the locally-defined tensor in (5.3) represent the components of the spin per particle.

The corresponding spin vector S^α related to the v^α congruence is

$$S^\alpha = \frac{1}{2}\eta^{\alpha\beta\gamma\delta}v_\beta \frac{\bar{\Psi}\sigma_{\gamma\delta}\Psi}{\bar{\Psi}\Psi} = S_0^\alpha + \hbar S_1^\alpha + \dots. \tag{5.4}$$

To order $\hbar^0$ we have

$$S_0^\alpha = \tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}u_\beta \overline{b_0}\sigma_{\gamma\delta}b_0 \tag{5.5}$$

with

$$(-S_0^\alpha S_{0\alpha})^{1/2} = 1 \tag{5.6}$$

and

$$\overline{b_0}\sigma^{\alpha\beta}b_0 = \eta_{\alpha\beta\gamma\delta}u^\gamma S_0^\delta. \tag{5.7}$$

The spin motion is now an immediate consequence of the propagation equations (3.19) with equations (2.5a) and (2.13):

$$(\overline{b_0}\sigma_{\alpha\beta}b_0)_{;\epsilon}u^\epsilon = 0, \qquad S^\alpha_{0;\epsilon}u^\epsilon = 0. \tag{5.8}$$

Accordingly we find for the spin motion a parallel propagation along the v^α congruence

$$\left(\frac{\bar{\Psi}\sigma_{\alpha\beta}\Psi}{\bar{\Psi}\Psi}\right)_{;\epsilon} v^\epsilon = \mathrm{O}(\hbar), \qquad S^\alpha{}_{;\epsilon}v^\epsilon = \mathrm{O}(\hbar). \tag{5.9}$$

This quantum mechanical result is to be compared with the classical equation (1.1b).

6. Trajectories

The congruence of the streamlines formed by the convection current will in general be geodesic only to order zero in $\hbar$. Corrections of higher order in $\hbar$ represent the influence of the spin on the trajectories. We obtain v^α in normalising j_c^α of equation (4.9).

$$v^\alpha = u^\alpha + (\hbar/2m\mathrm{i})(\overline{b_0}^{;\alpha}b_0 - \overline{b_0}b_0{}^{;\alpha}) + \mathrm{O}(\hbar^2). \tag{6.1}$$

Note that the term in equation (4.9) which contains a_1 does not appear in the coefficient of $\hbar$. This enables the determination of the deviation from the geodesic behaviour

$$v_{\alpha;\epsilon}v^{\epsilon} = 2v_{1[\alpha;\epsilon]}u^{\epsilon} + \mathrm{O}(\hbar^2) \tag{6.2}$$

already within this order. v_1^{α} is the term of order $\hbar$ in equation (6.1). Using the commutator of the covariant spin derivative

$$\chi_{[;\alpha;\beta]} = -(\mathrm{i}/8)R_{\alpha\beta\gamma\delta}\sigma^{\gamma\delta}\chi \tag{6.3}$$

we finally obtain as the generalised *force equation* for the v^{α} congruence

$$mv^{\alpha}{}_{;\epsilon}v^{\epsilon} = (\hbar/2)(1/2)R_{\alpha\beta\gamma\delta}\overline{b_0}\sigma^{\gamma\delta}b_0u^{\beta} + \mathrm{O}(\hbar^2). \tag{6.4}$$

This describes the deviation from the geodesic behaviour due to the coupling of the spin density to the curvature. It is to be compared with (1.1a).

It is of interest to note the result which would be obtained without Gordon decomposition, if we choose the normalised vector v^{α} to be parallel to the probability current j^{α} instead of to the convection current j_c^{α}. In this case using equation (4.10) we find equation (6.4) with the additional term

$$\begin{aligned} 2m\hbar\left(\frac{j_{M1[\alpha}}{f^2}\right)_{;\epsilon]}u^{\epsilon} = {} & -\hbar\eta_{\alpha}{}^{\beta}{}_{\kappa\lambda}u^{\kappa}\{\theta_{,\beta}S_0^{\lambda} + (\hat{\sigma}^{\epsilon}{}_{\beta} + (\theta/3)h^{\epsilon}{}_{\beta})[(\ln f^2)_{;\epsilon}S_0^{\lambda} + S_{0;\epsilon}^{\lambda}]\} \\ & -\hbar R^{*}_{\epsilon\mu\alpha\kappa}S_0^{\mu}u^{\epsilon}u^{\kappa} \end{aligned} \tag{6.5}$$

on the right-hand side. Recalling the interpretation of the magnetisation current j_M^{α} as curl of the spin density, it is plausible that its propagation is determined by space-like derivatives of the u^{α} field as well. This introduces the kinematical quantities of the u^{α} congruence.

7. Discussion

7.1.

The quantum mechanical equations (5.9) and (6.4) reflect the classical equations (1.1) if one identifies P^{α} with mj_c^{α}. In the classical approach there is in general no restriction of the time-like worldline $z^{\alpha}(x)$ with the tangent vector V^{α}. On the other hand, the quantum mechanical considerations refer to an infinitesimal tube around an arbitrary worldline taken out of a congruence of streamlines. Therefore to enable a comparison we must also specialise $z^{\alpha}(x)$ to the centre-of-mass line. In this case P^{α} is very nearly parallel to V^{α} under most circumstances. This completes the comparison as far as extended bodies are concerned. For pole–dipole point particles the difference between P^{α} and V^{α} is interpreted as reflecting the *Zitterbewegung*. Because of the WKB approach we are essentially restricted to positive-energy solutions. Therefore no such *Zitterbewegung* is to be expected.

7.2.

The identification of the dynamically defined P^{α} with mj_c^{α} instead of mj^{α} has already been partly justified in § 4. We add that because of equation (4.7) it is mj_c^{α} which according to its structure (4.4) represents a momentum density. Note that because of equations (5.1) and (6.1) we have a physical density on the right-hand side of equation (6.4) as well.

7.3.

It is typical that the force equations (6.4) which describe the influence of the spin were obtained by using second derivatives of the spinor field. For point particles the equation based on second derivatives already represents tidal forces. That the non-geodesic behaviour of Dirac particles results from a non-local interaction with the metric field reflects the fact that Dirac particles appear to be extended over a domain of linear dimensions $\hbar/m$ if one excludes negative-energy solutions. This has been done above by restriction to the first order in $\hbar$ of a WKB expansion.

7.4.

It is remarkable that there is a complete correspondence between the classical formula (1.1*a*) and the quantum mechanical result (6.4), not only with regard to the overall structure of the equations but also including the numerical factors. This result can be taken as a confirmation that for arbitrary metric fields to the first order of a WKB expansion of the fermion field the *gyro-gravitational factors* of the classical angular momentum and of the intrinsic quantum mechanical spin agree. This is in contrast to the gyromagnetic factors in external electromagnetic fields. Accordingly for fermions all types of angular momentum couple with the same ratio to the 'magnetic type' gravitational components contained in a general metric field. That the gyro-gravitational factors agree has also been shown for certain weak gravitational fields in the non-relativistic approximation using the Foldy–Wouthuysen representation by De Oliveira and Tiomno (1962).

7.5. Summary

Using a WKB scheme and a Gordon decomposition we have shown for arbitrary space-times in a generally covariant and observer-independent way that the equations of motion for the spin and the spin-induced non-geodesic behaviour of the trajectories are in total accordance with the corresponding classical equations including numerical factors.

Appendix

We give a brief survey of the previous literature and point out differences from the above approach.

Lawrence (1970) also treats the influence of spin on the trajectories of fermions by means of a WKB type approximation. His approach differs from our treatment in the following points: Lawrence considers only the approximation of linearised gravitational fields in flat space–time. His calculation is based on the iterated Dirac equation, and he claims that this is necessary to obtain a spin-induced non-geodesic behaviour of the trajectories. This was derived above from the non-iterated Dirac equation. A serious objection to his calculation seems to be that he makes use of the condition $\bar{\Psi}\Psi = 1$.

Equation (3.15) shows that this amounts to the restriction that the trajectories should be expansion-free, which is in general not the case.

Ehrenfest's equations demonstrate that in non-relativistic quantum mechanics equations of motion can most directly be derived in the Heisenberg picture. In curved space–time the corresponding approach would have the disadvantage that the results would depend on the observer congruence to which the time-like derivative is related. To avoid this dependence, a number of authors (Wong 1972, Rumpf 1979, Drechsler 1979) have introduced a generalised Heisenberg picture. They enlarge the space of dynamical variables by introducing t and $-\mathrm{i}\hbar\, \partial/\partial t$ as an additional pair of conjugate dynamical operators. In addition a generalised time dependence is introduced as the s variation

$$\mathrm{i}\hbar\dot{O} = \mathrm{i}\hbar\, \mathrm{d}O/\mathrm{d}s = [O, \hat{H}], \qquad \hat{H} = -\gamma^{\mu}P_{\mu}$$

so that a $4+1$ formalism results. Operators are then replaced according to an idea of Corben (1968) by

$$O' = O + (\mathrm{i}\hbar/2m)O$$

which allows an ordering in powers of $\hbar$. The equations for the O' can finally be compared with the classical equations. This $4+1$ formalism resembles closely the proper time technique of Fock (1937) which was popularised by Schwinger (1951). But in the latter cases the formalism only serves as a method of obtaining rigorous solutions which are then interpreted within the original context.

Although successful from a formal point of view, this $4+1$ redefinition of dynamics has the weakness that the relation to the usual physical interpretations based on expectation values has not yet been satisfactorily established. In this sense the procedure is incomplete. Wong (1972) shows that to establish an equivalence with the original Heisenberg picture one has to assume that expectation values of operator products may be replaced by products of expectation values.

The *Zitterbewegung* does not appear if one changes to the mean position and mean spin operators of the Foldy–Wouthuysen representation. For the case of Dirac theory in linearised gravitational fields in flat space–time this has been done by Kannenberg (1977). He calculates relativistic corrections to the respective two-component Pauli equation. The structure obtained is considerably more complex than equation (1.1). Furthermore, because the procedure is based on a Hermitian Hamiltonian related to a particular time-like coordinate, manifest general covariance is lost, which makes a physical interpretation of the terms obtained rather difficult.

Finally we must mention Barker and O'Connel (1970) who studied in flat space–time the gravitational two-body interaction due to graviton exchange. By replacing the spin terms in the corresponding Hamiltonian in an appropriate way by classical quantities, by dropping the contact terms and restricting consideration to order $(v/c)^2$, the classical results for the motion of a gyroscope can be obtained.

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